Generalised discriminants, deformed quantum Calogero-Moser system and Jack polynomials

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Abstract. It is shown that the deformed Calogero-Moser-Sutherland (CMS) operators can be described as the restrictions on certain affine subvarieties (called generalised discriminants) of the usual CMS operators for infinite number of particles. The ideals of these varieties are shown to be generated by the Jack symmetric functions related to the Young diagrams with special geometry. A general structure of the ideals which are invariant under the action of the quantum CMS integrals is discussed in this context. The shifted super-Jack polynomials are introduced and combinatorial formulas for them and for super-Jack polynomials are given.

1. Introduction

The primary goal of this paper is to explain the algebraic nature of integrability of the deformed Calogero-Moser-Sutherland (CMS) operators

$$
\begin{aligned}
L_{n,m,\theta} = &-\left(\frac{\partial^2}{\partial {x_1}^2}+\cdots+\frac{\partial^2}{\partial {x_n}^2}\right)-k\left(\frac{\partial^2}{\partial {y_1}^2}+\cdots+\frac{\partial^2}{\partial {y_m}^2}\right)\\
&+\sum_{i<j}^{n}\frac{2k(k+1)}{\sin^2(x_i-x_j)}+\sum_{i<j}^{m}\frac{2(k^{-1}+1)}{\sin^2(y_i-y_j)}+\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{2(k+1)}{\sin^2(x_i-y_j)}
\end{aligned}
\qquad (1)
$$

For $m=1$ these operators have been first introduced in [1, 2], for general m they were considered in [3, 4]. In [5] we have suggested a general construction of the deformed CMS operators related to Lie superalgebras, which in the case of Lie superalgebra $sl(n|m)$ leads to the operators (1). Unfortunately this relation with Lie superalgebras itself does not supply the integrability of this problem, which was proved in [5] by direct construction of the quantum integrals.

In this paper we will present a more conceptual proof of the integrability of (1) by showing that (after some gauge transformation and change of variables) the deformed CMS operators can be described as the *restriction*

of the usual CMS operators for infinite number of particles onto certain subvarieties of Macdonald variety called generalised discriminants. Note that the restriction of a differential operator onto a submanifold is possible only under very special circumstances. In case of the algebraic subvariety this means that the corresponding ideal must be invariant under the action of the operator.

For the proof we use the theory of Jack polynomials [6], [7] and the theory of shifted Jack polynomials developed recently by Knop, Sahi, Okounkov and Olshanski [8, 9, 10]. We have been partially inspired by a recent very interesting paper [11] by B. Feigin, Jimbo, Miwa and Mukhin, where certain ideals in the rings of symmetric polynomials were described in terms of Jack polynomials.

The structure of the paper is following. First we review the basic facts from the theory of Jack and shifted Jack polynomials and from the theory of Cherednik-Dunkl operators.

In section 5 we introduce the generalised discriminants and prove our main result. In section 6 we show that the quantum integrals we have constructed in [5] can also be described as the restrictions of certain integrals of the usual CMS problem. The notion of the shifted super-Jack polynomials naturally appears in this relation.

Section 7 is devoted to the description of the ideals in the algebra of symmetric functions which are invariant under the action of the quantum integrals of the CMS system. We show that the rectangular Young diagrams related to generalised discriminants play a very special role here.

In the last section we give combinatorial formulas for the super-Jack and shifted super-Jack polynomials generalising the results of Stanley, Okounkov and Olshanski [7, 10, 12].

2. Symmetric functions and Jack polynomials

In this section we recall some general facts about symmetric functions and Jack polynomials mainly following Macdonald's book [6]. It would be convenient for us to use instead of the parameter α in Macdonald's notations of Jack polynomials the parameter

$$\theta = \frac{1}{\alpha}$$

(cf. [10]). It is different from the parameter k used in our previous work [5] by sign change:

$$\theta = -k.$$

Let $P_N = \mathbb{C}[x_1, \dots, x_N]$ be the polynomial algebra in N independent variables and $\Lambda_N \subset P_N$ be the subalgebra of symmetric polynomials.

A *partition* is any sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r \dots)$$

of nonnegative integers in decreasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \ldots$$

containing only finitely many nonzero terms. The number of nonzero terms in λ is the *length* of λ denoted by $l(\lambda)$. The sum $\mid \lambda \mid = \lambda_1 + \lambda_2 + \ldots$ is called the *weight* of λ. The set of all partitions of weight N is denoted by $\mathcal{P}_N$.

On this set there is a natural involution: in the standard diagrammatic representation [6] it corresponds to the transposition (reflection in the main diagonal). The image of a partition λ under this involution is called the *conjugate* of λ and denoted by λ'. This involution will play an essential role in our paper.

Partitions can be used to label the bases in the symmetric algebra Λ_N. There are several important bases in Λ_N.

1) *Monomial symmetric polynomials* $m_\lambda, \lambda \in \mathcal{P}_N$ are defined by

$$m_\lambda(x_1, \ldots, x_N) = \sum x_1^{a_1} x_2^{a_2} \ldots x_N^{a_N}$$

summed over all distinct permutations a of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$.

2) *Elementary symmetric polynomials* are defined by

$$\sum_{k=o}^{\infty} e_k t^k = \prod_{i \geq 1} (1 + x_i t).$$

For each partition λ we define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$$

3) Similarly *complete symmetric polynomials* are defined by

$$\sum_{k=o}^{\infty} h_k t^k = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

and

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots.$$

4) Finally the most convenient for us will be the *power sums*

$$p_k = x_1^k + x_2^k + \ldots,$$

where again for any partition λ

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$$

It is well-known [6] that each of these sets of functions with $l(\lambda) \leq N$ form a basis in Λ_N.

We will need the following infinite dimensional versions of both P_N and Λ_N. Let $M \leq N$ and $\varphi_{N,M} : P_N \longrightarrow P_M$ be the homomorphism which sends each of $x_{M+1}, \ldots, x_N$ to zero and other x_i to themselves. It is clear that $\varphi_{N,M}(\Lambda_N) = \Lambda_M$ so we can consider the inverse limits in the category of graded algebras

$$P = \varprojlim P_N, \quad \Lambda = \varprojlim \Lambda_N.$$

This means that

$$P = \oplus_{r=0}^{\infty} P^r, \quad P^r = \varprojlim P_N^r$$

$$\Lambda = \oplus_{r=0}^{\infty} \Lambda^r, \quad \Lambda^r = \varprojlim \Lambda_N^r$$

where P_N^r, Λ_N^r are the homogeneous components of P_N, Λ_N of degree r. The elements of Λ are called *symmetric functions.*

Since for any partition λ

$$\varphi_{N,M}(m_\lambda(x_1, \dots, x_N)) = m_\lambda(x_1, \dots, x_M)$$

(and similarly for the polynomials h, e, p) we can define the symmetric functions $m_\lambda, h_\lambda, e_\lambda, p_\lambda$.

Another important example of symmetric functions are *Jack polynomials.* We give here their definition in the form most suitable for us.

Recall that on the set of partitions $\mathcal{P}_N$ there is the following *dominance partial ordering*: we write $\mu \le \lambda$ if for all $i \ge 1$

$$\mu_1 + \mu_2 + \dots + \mu_i \le \lambda_1 + \lambda_2 + \dots + \lambda_i.$$

Consider the following *CMS operator*

$$\mathcal{L}_\theta^{(N)} = \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i} \right)^2 + \theta \sum_{1 \le i < j \le N} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) - \theta (N-1) \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i} \right)^2 + 2\theta \sum_{i \ne j} \frac{x_i x_j}{x_i - x_j} \frac{\partial}{\partial x_i}. \tag{2}$$

It coincides with the standard (trigonometric) Calogero-Moser-Sutherland operator [13, 14] if we change the gauge and use exponential coordinates. It is related to the so-called Laplace-Beltrami operator

$$\Box_N^\alpha = \frac{\alpha}{2} \sum_{i=1}^N x_i^2 \left(\frac{\partial}{\partial x_i} \right)^2 + \sum_{i \ne j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i} - \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}$$

used by Macdonald [6] by a simple formula

$$\mathcal{L}_\theta^{(N)} = 2\theta \Box_N^{\frac{1}{\theta}} - \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}.$$

An important property of the CMS operator is its *stability* under the change of N : the following diagram is commutative

$$\begin{array}{ccc} \Lambda_N & \xrightarrow{\mathcal{L}_\theta^{(N)}} & \Lambda_N \\ \downarrow \varphi_{N,M} & & \downarrow \varphi_{N,M} \\ \Lambda_M & \xrightarrow{\mathcal{L}_\theta^{(M)}} & \Lambda_M \end{array}$$

(see [6], example 3 on the page 326). This allows us to define the CMS operator $\mathcal{L}_\theta$ on the space of symmetric functions Λ as the limit of $\mathcal{L}_\theta^{(N)}$.

Theorem-Definition. *If θ is not negative rational number or zero than for any partition λ , $l(\lambda) \leq N$ there is a unique polynomial $P_\lambda(x, \theta) \in \Lambda_N$ (called Jack polynomial) such that*

1) $P_\lambda(x, \theta) = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu$, where $u_{\lambda\mu} \in \mathbb{C}$

2) $P_\lambda(x, \theta)$ is an eigenfunction of the CMS operator $\mathcal{L}_\theta^{(N)}$.

To prove this one can check that the operator $\mathcal{L}_\theta^{(N)}$ has an upper triangular matrix in the monomial basis m_μ:

$$\mathcal{L}_\theta^{(N)}(m_\lambda) = \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu$$

where the coefficients $c_{\lambda\mu}$ can be described explicitly (see [6], page 327 and [15]). In particular

$$c_{\lambda\lambda} = \sum_{i=1}^{N} \lambda_i^2 - 2\theta \sum_{i=1}^{N} (i-1)\lambda_i = 2n(\lambda') - 2\theta n(\lambda) + |\lambda|$$

where $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$. It is easy to see that if $\lambda > \mu$ then $n(\lambda) < n(\mu)$ and $n(\lambda') > n(\mu')$. So we have

$$c_{\lambda\lambda} - c_{\mu\mu} = 2[n(\lambda) - n(\mu)] + 2\theta[n(\mu') - n(\lambda')]$$

Since θ is not a negative rational we see that $c_{\lambda\lambda} \neq c_{\mu\mu}$ if $\lambda > \mu$. This implies the claim.

From the stability of the CMS operators it follows that

$$\varphi_{N,M}(P_\lambda(x_1, \ldots, x_N)) = P_\lambda(x_1, \ldots, x_M)$$

so we have correctly defined Jack symmetric functions $P_\lambda(x, \theta) \in \Lambda$ which are the eigenfunctions of the CMS operator $\mathcal{L}_\theta$.

3. Shifted symmetric functions and shifted Jack polynomials.

We discuss now the so-called *shifted Jack polynomials* introduced recently by Knop, Sahi, Okounkov and Olshanski [8, 9, 10, 12]. For a nice review of the theory of shifted symmetric functions and its role in the representation theory of the symmetric group we refer to [16].

Let us denote by $\Lambda_{N,\theta}$ the algebra of polynomials $f(x_1, \ldots, x_N)$ which are symmetric in the shifted variables $x_i + \theta(1-i)$. This algebra has the filtration by the degree of polynomials:

$$(\Lambda_{N,\theta})_0 \subset (\Lambda_{N,\theta})_1 \subset \cdots \subset (\Lambda_{N,\theta})_r \subset \ldots$$

We have the following shifted analog of power sums:

$$p_r^*(x_1, \ldots, x_N, \theta) = \sum_{i=1}^{N} \left[(x_i + \theta(1-i))^r - (\theta(1-i))^r \right]. \tag{3}$$

The polynomials

$$p_\lambda^*(x, \theta) = p_{\lambda_1}^*(x, \theta) p_{\lambda_2}^*(x, \theta) \ldots$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r \dots)$ is a partition of N form a basis in $\Lambda_{N,\theta}$. Since $\varphi_{N,M}(\Lambda_{N,\theta}) = \Lambda_{M,\theta}$ one can consider the inverse limit

$$\Lambda_\theta = \lim_{\longleftarrow} \Lambda_{N,\theta}$$

in the category of filtered algebras:

$$\Lambda_\theta = \bigcup_{r=0}^{\infty} (\Lambda_\theta)_r, \quad (\Lambda_\theta)_r = \lim_{\longleftarrow} (\Lambda_{N,\theta})_r.$$

The algebra Λ_θ is called the algebra of *shifted symmetric functions.*

The *shifted Jack polynomials* can be defined in the following way (see [9],[10]). Let us introduce the following function on the set of partitions

$$H(\lambda, \theta) = \prod_{\square \in \lambda} (c'_\theta(\square) + 1) \tag{4}$$

Here we use the diagrammatic representation of the partitions using the squares [6] and to each square $\square = (i, j)$ we prescribe the numbers

$$c'_\theta(\square) = \lambda_i - j + \theta(\lambda'_j - i). \tag{5}$$

Theorem-Definition [9, 10]**.** *Let λ be a partition with $\lambda_{N+1} = 0$. There exists a unique shifted symmetric polynomial $P^*_\lambda(x, \theta) \in \Lambda_{N,\theta}$ (called shifted Jack polynomial) such that* $\deg P_\lambda \le |\lambda|$ *and*

$$P^*_\lambda(\mu, \theta) = \begin{cases} H(\lambda), & \mu = \lambda \\ 0, & |\mu| \le |\lambda|, \mu \ne \lambda, \mu_{N+1} = 0 \end{cases}$$

Here and later throughout the paper by $P(\lambda)$ for a polynomial $P(x_1, \dots, x_N)$ and a partition $\lambda = (\lambda_1, \dots, \lambda_N)$ we mean $P(\lambda_1, \dots, \lambda_N)$. It is actually a very useful idea to view the shifted symmetric polynomials as the functions on the partitions (see e.g. [16]).

It is easy to see from the definition that the shifted Jack polynomials are stable:

$$\varphi_{N,M}(P^*_\lambda(x_1, \dots, x_N, \theta)) = P^*_\lambda(x_1, \dots, x_M, \theta),$$

so we can define the shifted Jack function $P^*_\lambda(x, \theta)$ as an element of Λ_θ.

Knop and Sahi [9] proved that the shifted Jack polynomials satisfy the following so-called *Extra Vanishing Property*:

$$P^*_\lambda(\mu, \theta) = 0 \tag{6}$$

unless the diagram of λ is a subset of the diagram of μ and that $P^*_\lambda(x, \theta)$ is the usual Jack polynomial $P_\lambda(x, \theta)$ plus lower order terms.

We will need the following *duality* property of the shifted Jack polynomials:

$$P^*_\lambda(\mu', \theta) = \frac{H(\lambda, \theta)}{H(\lambda', 1/\theta)} P^*_{\lambda'}(\mu, 1/\theta). \tag{7}$$

For shifted Macdonald polynomials this was proved by Okounkov [12]. We present here an independent proof for Jack polynomials based on the Bernoulli sums.

Consider the following natural *conjugation homomorphism*:

$$(\omega^*(f))(\lambda) = f(\lambda') \tag{8}$$

Proposition 1. *The conjugation homomorphism maps the algebra of shifted symmetric functions Λ_θ into the algebra $\Lambda_{1/\theta}$.*

Proof. Let us introduce the following *Bernoulli sums*:

$$b_k(x_1, \ldots, x_N, \theta) = \sum_{i=1}^{N} \left[B_k(x_i + \theta(1-i)) - B_k(\theta(1-i))\right], \tag{9}$$

where $B_k(x)$ are the classical Bernoulli polynomials. Recall that $B_k(x)$ can be defined through the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and satisfy the property

$$B_k(x+1) - B_k(x) = kx^{k-1}$$

or, more generally

$$B_k(x+l) - B_k(x) = k \sum_{i=1}^{l} (x+i-1)^{k-1}. \tag{10}$$

It is easy to see that the Bernoulli sums are stable:

$$\varphi_{N,M}(b_k(x_1, \ldots, x_N, \theta)) = b_k(x_1, \ldots, x_M, \theta),$$

so one can define the *shifted symmetric Bernoulli functions* $b_k(x, \theta) \in \Lambda_\theta$.

Lemma 1. *Bernoulli functions satisfy the following symmetry:*

$$b_k(\lambda', \theta) = (-\theta)^{k-1} b_k(\lambda, 1/\theta). \tag{11}$$

The proof is a straightforward check:

$$b_k(\lambda', \theta) = \sum_{j=1}^{l(\lambda')} \left[B_k(\lambda'_j + \theta(1-j)) - B_k(\theta(1-j))\right] =$$

$$k \sum_{j=1}^{l(\lambda')} \sum_{i=1}^{\lambda'_j} [\theta(1-j) + i - 1]^{k-1} =$$

$$k \sum_{\square \in \lambda} [\theta(1-j) + i - 1]^{k-1} = k(-\theta)^{k-1} \sum_{\square \in \lambda} [1/\theta(1-i) + j - 1]^{k-1}$$

$$k(-\theta)^{k-1} \sum_{i=1}^{l(\lambda)} \sum_{i=1}^{\lambda_i} [1/\theta(1-i) + j - 1]^{k-1} =$$

$$(-\theta)^{k-1} \sum_{i=1}^{l(\lambda)} [B_k(\lambda_i + 1/\theta(1-i)) - B_k(\theta(1-i))] = (-\theta)^{k-1} b_k(\lambda, 1/\theta)$$

We have used here the property (10) of the Bernoulli polynomials.

Since Bernoulli sums $b_k(x,\theta)$ generate the algebra Λ_θ the proposition now follows. Combining this with the definition of the shifted Jack polynomials we have the duality property (7).

4. Cherednik - Dunkl operators and Harish-Chandra homomorphism

In this section we present the basic facts about Cherednik-Dunkl operators. For the details we refer to Opdam's review [18].

By *Cherednik - Dunkl operators* we mean the following difference-differential operators

$$D_{i,N} = x_i \frac{\partial}{\partial x_i} + \theta \sum_{j \neq i} \frac{x_{\max\{i,j\}}}{x_i - x_j}(1 - \sigma_{ij}), \quad i,j \leq N, \tag{12}$$

where σ_{ij} is acting on the function $\phi(x_1, \dots x_N)$ by permutation of i-th and j-th coordinates. If $i > N$ we assume that $D_{i,N} = 0$. We should warn the reader that our definition is different from the usual one

$$\hat{D}_{i,N} = x_i \frac{\partial}{\partial x_i} + \theta \sum_{j \neq i} \frac{x_{\max\{i,j\}}}{x_i - x_j}(1 - \sigma_{ij}) + \theta(N - i),$$

(see e.g. [18]) by a shift. This shift is necessary for the stability of $D_{i,N}$ and explains the relations with the theory of shifted symmetric functions which we discuss below.

The first important property of the Cherednik - Dunkl operators is that they commute with each other:

$$[D_{i,N}, D_{j,N}] = 0.$$

This means that one can substitute them in any polynomial P in N variables without ordering problems.

The second property is that if one does this for a shifted symmetric polynomial $f \in \Lambda_{N,\theta}$ then the corresponding operator $f(D_{1,N} \dots D_{N,N})$ leaves the algebra of symmetric polynomials Λ_N invariant:

$$f(D_{1,N} \dots D_{N,N}) : \Lambda_N \to \Lambda_N.$$

The restriction of the operator $f(D_{1,N} \dots D_{N,N})$ on the algebra Λ_N is given by some differential operator, which we will denote as $\mathcal{L}^f_{N,\theta}$. A formula for this operator can be found by moving all the permutation operators in $f(D_{1,N} \dots D_{N,N})$ to the right using natural commutation relations and then erasing them.

One can check that if we apply this operation to the shifted square sum $p_2^*(x_1, \dots, x_N, \theta) = \sum_{i=1}^N \left[(x_i + \theta(1-i))^2 - (\theta(1-i))^2\right]$ we arrive at the CMS operator (2). Thus all the operators $\mathcal{L}^f_{N,\theta}$ are actually the quantum integrals of the CMS problem. The Jack polynomials are the joint eigenfunctions of all these operators: if $P_\lambda(x,\theta)$ is the Jack polynomial corresponding

to a partition λ of length N then

$$\mathcal{L}^f_{N,\theta} P_\lambda(x,\theta) = f(\lambda_1, \lambda_2, \dots, \lambda_N) P_\lambda(x, \theta.) \tag{13}$$

This allows us to define a homomorphism (which is actually a monomorphism) $\chi : f \to \mathcal{L}^f_{N,\theta}$ from the algebra $\Lambda_{N,\theta}$ to the algebra of differential operators. Let us denote by $\mathcal{D}(N,\theta)$ the image of χ. The inverse homomorphism

$$\chi^{-1} : \mathcal{D}(N,\theta) \longrightarrow \Lambda_{N,\theta}$$

is called the *Harish-Chandra isomorphism.* It can be defined by the action on the Jack polynomials: the image of $\mathcal{L} \in \mathcal{D}(N,\theta)$ is a polynomial $f = f_{\mathcal{L}} \in \Lambda_{N,\theta}$ such that

$$\mathcal{L} P_\lambda(x,\theta) = f(\lambda) P_\lambda(x,\theta).$$

One can check that the Cherednik-Dunkl operators $D_{i,N}$ are stable (note that this is not true for the shifted operators $\hat{D}_{i,N}$): the diagram

$$\begin{array}{ccc} P_N & \xrightarrow{D_{i,N}} & P_N \\ \downarrow \varphi_{N,M} & & \downarrow \varphi_{N,M} \\ P_M & \xrightarrow{D_{i,M}} & P_M \end{array}$$

is commutative for all $M \le N$ and $i \ge 1$. Similarly for any $f \in \Lambda_{N,\theta}$ and $g = \varphi_{N,M}(f), M \le N$ the following diagram is commutative:

$$\begin{array}{ccc} \Lambda_N & \xrightarrow{\mathcal{L}^f_{N,\theta}} & \Lambda_N \\ \downarrow \varphi_{N,M} & & \downarrow \varphi_{N,M} \\ \Lambda_M & \xrightarrow{\mathcal{L}^g_{M,\theta}} & \Lambda_M \end{array}$$

This allows us to define for any shifted symmetric function $f \in \Lambda_\theta$ a differential operator

$$\mathcal{L}^f_\theta : \Lambda \longrightarrow \Lambda$$

and the infinite dimensional version of the homomorphism χ. We will denote by $\mathcal{D}(\theta)$ the image of this homomorphism, which can be interpreted as the ring of quantum integrals of the (infinite-dimensional) CMS problem. The inverse (Harish-Chandra) homomorphism $\chi^{-1} : \mathcal{D}(\theta) \longrightarrow \Lambda_\theta$ can be described by the relation

$$\mathcal{L}^f_\theta P(\lambda,\theta) = f(\lambda) P(\lambda,\theta),$$

where now $f \in \Lambda_\theta$ and $P(\lambda,\theta)$ is Jack symmetric function.

5. Generalised discriminants and deformed CMS operators.

Now we are ready to formulate our main results. The central role in our construction will play the following algebra $\Lambda_{n,m,\theta}$ introduced in [5] (denoted there as $\Lambda^0_{n,m,\theta}$).

Let $P_{n,m} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ be the polynomial algebra in $n+m$ independent variables. Then $\Lambda_{n,m,\theta} \subset P_{n,m}$ is the subalgebra consisting

of polynomials which are symmetric in $x_1,\dots,x_n$ and $y_1,\dots,y_m$ separately and satisfy the conditions

$$\left(x_i\frac{\partial}{\partial x_i}+\theta y_j\frac{\partial}{\partial y_j}\right)f\equiv 0 \tag{14}$$

or, equivalently

$$\left(\frac{\partial}{\partial x_i}+\theta\frac{\partial}{\partial y_j}\right)f\equiv 0 \tag{15}$$

on each hyperplane $x_i-y_j=0$ for $i=1,\dots,n$ and $j=1,\dots,m$. It is shown in [5] that for generic θ (namely, if θ is not a negative rational or zero) $\Lambda_{n,m,\theta}$ coincides with its subalgebra $\mathcal{N}_{n,m,\theta}$ generated by the *deformed Newton sums*

$$p_r(x,y,\theta)=\sum_{i=1}^{n}x_i^r-\frac{1}{\theta}\sum_{j=1}^{m}y_j^r \tag{16}$$

which obviously belong to $\Lambda_{n,m,\theta}$ for all nonnegative integers r.

This algebra has appeared in [5] in relation with the following *deformed CMS operator*

$$\mathcal{L}_{n,m,\theta}=\sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2-\theta\sum_{j=1}^{m}\left(y_j\frac{\partial}{\partial y_j}\right)^2+\theta\sum_{1\le i<j\le n}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right)-$$

$$\sum_{1\le i<j\le m}\frac{y_i+y_j}{y_i-y_j}\left(y_i\frac{\partial}{\partial y_i}-y_j\frac{\partial}{\partial y_j}\right)-\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{x_i+y_j}{x_i-y_j}\left(x_i\frac{\partial}{\partial x_i}+\theta y_j\frac{\partial}{\partial y_j}\right)-$$

$$(\theta(n-1)-m)\left(\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}+\sum_{j=1}^{m}y_j\frac{\partial}{\partial x_j}\right) \tag{17}$$

Lemma 2. *The deformed CMS operator preserves the algebra $\Lambda_{n,m,\theta}$:*

$$\mathcal{L}_{n,m,\theta}:\Lambda_{n,m,\theta}\to\Lambda_{n,m,\theta} \tag{18}$$

The proof follows from a more general statement proved in the next section (see Theorem 5).

Let now Λ be the algebra of symmetric functions in infinite number of variables $z_1,z_2,\dots,$ $p_r(z)=z_1^r+z_2^r+\dots$ be the power sums, $P_\lambda(z,\theta)$ be the Jack polynomials (see Section 2 above).

Consider the following homomorphism φ from Λ to $\Lambda_{n,m,\theta}$ such, that

$$\varphi(p_r(z))=p_r(x,y,\theta).$$

Since $p_r(z)$ are free generators of Λ this determines φ uniquely. Such a homomorphism was first used by Kerov, Okounkov and Olshanski in [19].

Our central result can be formulated as follows. Let $\mathcal{L}_\theta$ be the usual CMS operator in infinite dimension.

Theorem 1. *The following diagram is commutative for all values of the parameter θ:*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathcal{L}_\theta} & \Lambda \\ \downarrow \varphi & & \downarrow \varphi \\ \Lambda_{n,m,\theta} & \xrightarrow{\mathcal{L}_{n,m,\theta}} & \Lambda_{n,m,\theta} \end{array} \tag{19}$$

Before going to the proof let us discuss the geometric meaning of this claim. For generic values of the parameter θ according to [5] the algebra $\Lambda_{n,m,\theta}$ is finitely generated, so we can introduce an affine algebraic variety

$$\mathcal{D}_{n,m,\theta} = Spec\,\Lambda_{n,m,\theta} = Spec\,\mathcal{N}_{n,m,\theta}.$$

We call this variety *generalised discriminant* because for special values $m = 1$ and $\theta = -1/2$ the algebra $\mathcal{N}_{n,1,-1/2}$ coincides with the algebra of functions on the standard discriminant variety $\mathcal{D}_{n+1}$ of the polynomials of degree $n+1$ having a multiple root. A more general case $\theta = -1/q$ corresponds to the stratum in the discriminant variety when one of the roots has a multiplicity at least q, so informally speaking the generalised discriminant $\mathcal{D}_{n,m,\theta}$ consists of "polynomials" with n simple roots and m roots of "multiplicity" $\mu = -\theta^{-1}$.

We understand that our terminology is not perfect since the term "discriminant" was used and generalised in many different ways (see e.g. well-known book [20] by Gelfand, Kapranov and Zelevinsky) but an alternative term "generalised coincident root loci" looks too long and not much better. We would like to mention that the problem of finding the algebraic equations defining the strata in the discriminants is non-trivial and goes back to Arthur Cayley [21] (see [22, 23, 24] for the recent results in this direction).

The map φ determines an embedding of $\mathcal{D}_{n,m,\theta}$ into $\mathcal{M} = Spec\,\Lambda$. We will call $\mathcal{M}$ *Macdonald variety* although strictly speaking it is defined only as an affine scheme since the algebra Λ is not finitely generated. Since the algebra $\Lambda_{n,m,\theta}$ is also not finitely generated for special positive rational θ (see [5]) the Macdonald variety is a proper space for the generalised discriminants to live in.

Notice that for negative rationals the algebra $\Lambda_{n,m,\theta}$ could be bigger than $\mathcal{N}_{n,m,\theta}$, so in general we should distinguish the variety $\mathcal{D}_{n,m,\theta} = Spec\,\Lambda_{n,m,\theta}$ and its embedding in $\mathcal{M}$ $\tilde{\mathcal{D}}_{n,m,\theta} = Spec\,\mathcal{N}_{n,m,\theta}$.

Corollary. *The deformed CMS operator (17) is the restriction of the usual CMS operator $\mathcal{L}_\theta$ on Macdonald variety onto the generalised discriminant subvariety $\mathcal{D}_{n,m,\theta}$.*

We should mention that the restriction of a differential operator $\mathcal{L}$ onto a subvariety is a very rare phenomenon. Indeed this is possible only when the ideal corresponding to this subvariety is $\mathcal{L}$-invariant. In our case the situation is even more peculiar because we have a finite-dimensional subvariety in infinite dimensions.

Proof of Theorem 1. Let us introduce the following function $\Pi \in \Lambda[[t_1, \dots, t_N]]$ which plays an important role in the theory of Jack polynomials (see [7]):

$$\Pi = \prod_{l=1}^{N} \prod_{i \geq l} (1 - z_i t_l)^{-\theta}.$$

Lemma 3. *The function Π satisfies the following properties:*

(i)

$$\mathcal{L}_{\theta,z}\Pi = \mathcal{L}_{\theta,t}^{(N)}\Pi, \tag{20}$$

where index z (resp. t) indicates the action of the CMS operator $\mathcal{L}_\theta$ on z (resp. t) variables

(ii)

$$\varphi(\Pi) = \prod_{l=1}^{N} \prod_{i=1}^{n} (1 - x_i t_l)^{-\theta} \prod_{j=1}^{m} (1 - y_j t_l) \tag{21}$$

(iii)

$$\varphi(\mathcal{L}_{\theta,z}\Pi) = \mathcal{L}_{n,m,\theta}\varphi(\Pi) \tag{22}$$

Proof. Introduce the notation $c_{il} = \frac{z_i t_l}{1 - z_i t_l}$. The following identities are easy to verify

$$\frac{z_j c_{il} - z_i c_{jl}}{z_i - z_j} = c_{il} c_{jl}, \quad \frac{t_l c_{ik} - t_k c_{il}}{t_k - z_l} = c_{il} c_{ik}, \quad z_i \frac{\partial}{\partial z_i}(c_{il}) = t_l \frac{\partial}{\partial t_l}(c_{il}) = c_{il}(c_{il}+1).$$

Therefore we have

$$\Pi^{-1} z_i \frac{\partial}{\partial z_i}(\Pi) = \theta \sum_l c_{il}; \quad \Pi^{-1} t_l \frac{\partial}{\partial t_l}(\Pi) = \theta \sum_i c_{il},$$

$$\Pi^{-1} \left(z_i \frac{\partial}{\partial z_i} \right)^2 (\Pi) = \theta^2 \left(\sum_l c_{il} \right)^2 + \theta \sum_l c_{il}(c_{il} + 1),$$

$$\Pi^{-1} \left(t_l \frac{\partial}{\partial t_l} \right)^2 (\Pi) = \theta^2 \left(\sum_i c_{il} \right)^2 + \theta \sum_i c_{il}(c_{il} + 1).$$

Therefore

$$\Pi^{-1} \mathcal{L}_{\theta,z}(\Pi) = \theta^2 \sum_i \left(\sum_l c_{il} \right)^2 + \theta \sum_{i,l} c_{il}(c_{il} + 1) + 2\theta^2 \sum_{i<j} \sum_l c_{il} c_{jl}$$

and

$$\Pi^{-1} \mathcal{L}_{\theta,t}^{(N)}(\Pi) = \theta^2 \sum_l \left(\sum_i c_{il} \right)^2 + \theta \sum_{l,i} c_{il}(c_{il} + 1) + 2\theta^2 \sum_{k<l} \sum_i c_{il} c_{ik},$$

which are equal. This proves the first formula (20).

To prove (21) let us first note that since φ is a homomorphism it is enough to consider the case $N = 1$ when we have only one variable t. Introduce now

the following automorphism σ_θ which is defined by his action on the power sums as follows:

$$\sigma_\theta(p_r(y)) = -\frac{1}{\theta}p_r(y). \tag{23}$$

Then we have

$$\varphi\left(\prod_i (1-z_it)^{-\theta}\right) = \prod_i (1-x_it)^{-\theta}\sigma_\theta\left(\prod_j (1-y_jt)^{-\theta}\right) =$$

$$\prod_i (1-x_it)^{-\theta}\sigma_\theta\left(\exp\log\prod_j (1-y_jt)^{-\theta}\right) =$$

$$\prod_i (1-x_it)^{-\theta}\exp\sigma_\theta\left(-\theta\sum_{j\geq 1}\frac{p_j(y)t^j}{j}\right) = \prod_i (1-x_it)^{-\theta}\exp\left(-\sum_{j\geq 1}\frac{p_j(y)t^j}{j}\right) =$$

$$\prod_i^n (1-x_it)^{-\theta}\prod_j^m (1-y_jt).$$

This proves the second part of Lemma 3.

To prove the last part introduce $a_{il} = \frac{x_it_l}{1-x_it_l}, b_{jl} = \frac{y_jt_l}{1-y_jt_l}$. We have

$$\varphi(\Pi)^{-1}x_i\frac{\partial}{\partial x_i}\varphi(\Pi) = \theta\sum_l a_{il}, \quad \varphi(\Pi)^{-1}y_j\frac{\partial}{\partial y_j}\varphi(\Pi) = -\sum_l b_{jl},$$

$$\varphi(\Pi)^{-1}t_l\frac{\partial}{\partial t_l}\varphi(\Pi) = \theta\sum_i a_{il} - \sum_j b_{jl},$$

$$\varphi(\Pi)^{-1}\left(x_i\frac{\partial}{\partial x_i}\right)^2\varphi(\Pi) = \theta^2\left(\sum_l a_{il}\right)^2 + \theta\left(\sum_l a_{il}(a_{il}+1)\right),$$

$$\varphi(\Pi)^{-1}\left(y_j\frac{\partial}{\partial y_j}\right)^2\varphi(\Pi) = \left(\sum_l b_{jl}\right)^2 - \left(\sum_l b_{jl}(b_{jl}+1)\right),$$

$$\varphi(\Pi)^{-1}\left(t_l\frac{\partial}{\partial t_l}\right)^2\varphi(\Pi) = \left(\theta\sum_i a_{il} - \sum_j b_{jl}\right)^2 + \theta\sum_i a_{il}(a_{il}+1) - \sum_l b_{jl}(b_{jl}+1).$$

Now taking into account the following identity

$$\frac{(x_i+y_j)(a_{il}-b_{jl})}{x_i-y_j} = a_{il}+b_{jl}+2a_{il}b_{jl}.$$

we can write

$$\varphi(\Pi)^{-1}\mathcal{L}_{n,m,\theta}\varphi(\Pi) = \theta^2\sum_{i=1}^n\left(\sum_l a_{il}\right)^2 + \theta\left(\sum_{i,l}a_{il}(a_{il}+1) - \sum_{j=1}^m\left(\sum_l b_{jl}\right)^2 + \sum_{j,l}b_{jl}(b_{jl}+1)\right)$$

$$-\theta\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l}(a_{il}+b_{jl}+2a_{il}b_{jl})+\theta^2\sum_{1\le i<j\le n}\sum_{l}(a_{il}+a_{jl}+2a_{il}a_{jl})$$

$$+\sum_{1\le i<j\le m}\sum_{l}(b_{il}+b_{jl}+2b_{il}b_{jl})-(\theta(n-1)-m)\left(\theta\sum_{i,l}a_{il}-\sum_{j,l}b_{jl}\right)$$

and

$$\varphi(\Pi)^{-1}\mathcal{L}_{2,t}^{(N)}\varphi(\Pi)=\sum_{l}\left(\theta\sum_{i}a_{il}-\sum_{j}b_{jl}\right)^2+\theta\sum_{i,l}a_{il}(a_{il}+1)-\sum_{j,l}b_{jl}(b_{jl}+1)+$$

$$2\theta\sum_{k<l}\left(\theta\sum_{i=1}^{n}a_{il}a_{ik}-\sum_{j=1}^{m}b_{jl}b_{jk}\right).$$

It is easy to check that the last two expressions are identical, so $\varphi(\Pi)^{-1}\mathcal{L}_{n,m,\theta}\varphi(\Pi)=\varphi(\Pi)^{-1}\mathcal{L}_{\theta,t}^{(N)}\varphi(\Pi)=\varphi(\Pi)^{-1}\varphi(\mathcal{L}_{\theta,t}^{(N)}(\Pi))=\varphi(\Pi)^{-1}\varphi(\mathcal{L}_{\theta,z}(\Pi))$. This completes the proof of Lemma 3.

To complete the proof of Theorem 1 we should show that the coefficients $g_\lambda(z,\theta)$ in the expansion of the function

$$\Pi=\prod_{i,j}(1-z_it_j)^{-\theta}=\sum_{\lambda}g_\lambda(z,\theta)m_\lambda(t)$$

generate Λ when we increase the number of variables t. But this follows from the relation (see [6], I.4)

$$\sigma_\theta\left(\prod_{i,j}(1-z_it_j)^{-\theta}\right)=\prod_{i,j}(1-z_it_j)=\sum_{\lambda}(-1)^{|\lambda|}e_\lambda(z)m_\lambda(t),$$

where e_λ and m_λ are the standard symmetric functions defined in section 2, and the fact that e_λ form a basis in Λ. Theorem 1 is proved.

We believe that a similar statement true for any quantum integral $\mathcal{L}_f\in\mathcal{D}(\theta)$ of the CMS problem, which is equivalent to the fact that the kernel of φ is generated by certain Jack polynomials. For generic values of the parameter θ this follows from the following Theorem 2.

Let us introduce the set of partitions $H_{n,m}$, which consists of the partitions λ such that $\lambda_{n+1}\le m$ or, in other words, whose diagrams are contained in the *fat* (n,m) - *hook* (see fig.1). Its complement we will denote as $\bar{H}_{n,m}$. It consists of the diagrams which contain the $(n+1)\times(m+1)$ rectangle.

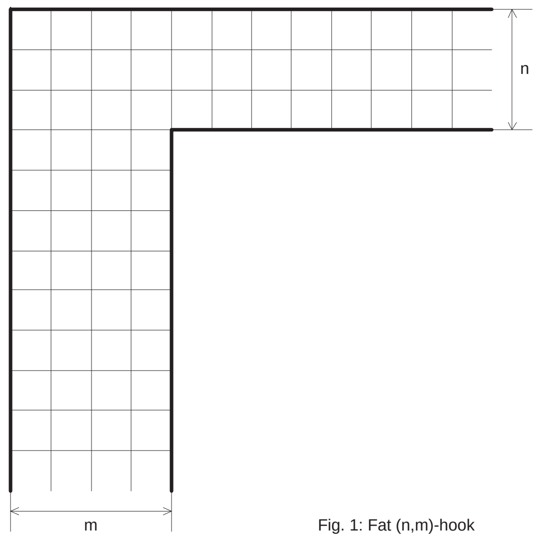

Fig. 1: Fat (n,m)-hook

Theorem 2. *If θ is not a negative rational number or zero, then $Ker\varphi$ is spanned by the Jack polynomials $P_\lambda(z,\theta)$ corresponding to the partitions which are not contained in the fat (n,m)-hook (or equivalently, which contain $(n+1)\times(m+1)$ rectangle).*

Proof. Notice first of all that if θ is not a negative rational number or zero the Jack polynomials $P_\lambda(z,\theta)$ are well defined, otherwise in general they may not exist.

Let us consider the following automorphism (see [7], [6], VI.10) ω_θ of algebra Λ :

$$\omega_\theta(p_r) = (-1)^{r-1}\theta p_r,$$

where p_r are the standard power sums. Then according to [6] (see page 380, formula (10.17)) we have

$$\omega_{\frac{1}{\theta}}(P_\lambda(z,\theta)) = \theta^{|\lambda|}\frac{H(\lambda,\theta)}{H(\lambda',1/\theta)}P_{\lambda'}(z,1/\theta),$$

where as before λ' is a partition conjugate to λ.

Let now $x=(x_1,x_2,\dots), y=(y_1,y_2,\dots,)$ be two infinite sequences variables. Then we have (see [6], page 345, formula $(7.9')$)

$$P_\lambda(x,y,\theta) = \sum_{\mu\subset\lambda} P_{\lambda/\mu}(x,\theta)P_\mu(y,\theta) \tag{24}$$

where $P_{\lambda/\mu}(z,\theta)$ are the *skew Jack functions* defined in [6] (see Chapter 6, § 7, 10) and $\mu\subset\lambda$ means that $\mu_i\le\lambda_i$ (or equivalently the diagram of μ is a subset of the diagram of λ).

Now consider the automorphism σ_θ which is combination of the automorphism $\omega_{1/\theta}$ with the change of sign $y_i\to -y_i$. The action of σ_θ on the power

sums is given by the formula (23):

$$\sigma_\theta(p_r(y)) = -\frac{1}{\theta} p_r(y).$$

If we apply this automorphism acting in y variables on both sides of the formula (24) and put all the variables x and y except the first n and m of them to zero we get

$$\varphi(P_\lambda(z,\theta)) = \sum_{\mu\subset\lambda} (-1)^{|\mu|} P_{\lambda/\mu}(x,\theta) \frac{H(\mu,\theta)}{\theta^{|\mu|} H(\mu', 1/\theta)} P_{\mu'}(y,\theta^{-1}). \tag{25}$$

Now let us assume that λ is not contained in the fat (n,m)-hook, then $\lambda'_{m+1} > n$. We have two possibilities: $\mu'_{m+1} > 0$ or $\mu'_{m+1} = 0$. In the first case we have $P_\mu(y_1,\dots,y_m,\theta^{-1}) = 0$ while in the second case $\lambda'_{m+1} - \mu'_{m+1} > n$, so according to [6] (page 347, formula (7.15)) the skew function $P_{\lambda/\mu}(x_1,\dots,x_n,\theta) = 0$. Thus we have shown that the Jack polynomials $P_\lambda(z,\theta)$ with $\lambda \in \bar{H}_{n,m}$ belong to the kernel of φ.

To prove that they actually generate the kernel consider the image of the Jack polynomials $P_\lambda(z,\theta)$ with $\lambda \in H_{n,m}$. From the formula (25) it follows that the leading term in lexicographic order of $\varphi(P_\lambda(z,\theta))$ has a form

$$(-1)^{\lambda_{n+1}+\lambda_{n+2}+\dots} x_1^{\lambda_1} \dots x_n^{\lambda_1} y_1^{<\lambda'_1 - n>} \dots y_m^{<\lambda'_m - n>},$$

where $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is the partition conjugate to λ and $< a >= \frac{a+|a|}{2} = \max(0,a)$. From the definition of φ it follows that $\varphi(P_\lambda(z,\theta)) \in \Lambda_{n,m,\theta}$. It is clear that all these polynomials corresponding to the diagrams contained in the fat hook are linearly independent. Theorem 2 is proved.

Notice that we have also shown that for the generic θ the polynomials

$$SP_\lambda(x,y,\theta) = \varphi(P_\lambda(z,\theta)), \quad \lambda \in H_{n,m} \tag{26}$$

form a linear basis in $\Lambda_{n,m,\theta}$. These polynomials are called *super-Jack polynomials* (see [25]).

Remark 1. A natural question is what are the algebraic generators of the algebra $\Lambda_{n,m,\theta}$. One can show using Theorem 2 that for generic θ the first $mn+m+n$ deformed Newton sums (16) $p_1(x,y,\theta),\dots,p_{mn+m+n}(x,y,\theta)$ (or equivalently the super-Jack polynomials $SP_\lambda(x,y,\theta)$ corresponding to the Young diagrams with one row of length less or equal than $mn+m+n$) generate this algebra. The number $mn+m+n$ here is the area of the rectangle $(n+1)\times(m+1)$ minus 1. One can use Theorem 2 to produce the relations between the generators (which are the equations determining the generalised discriminants) but a satisfactory description of them is still unknown. The fact that $\Lambda_{n,m,\theta}$ for generic θ is finitely generated was proved earlier in [5].

Remark 2. For special θ of the form $-\frac{1}{q}$ the homomorphism φ can be passed through the finite dimenion $N = n + mq$: $\varphi = \phi \circ \varphi_N$, where $\varphi_N : \Lambda \to \Lambda_N$ is the standard map (all z_i except N go to zero), and $\phi : \Lambda_N \to \Lambda_{n,m,\theta}$ is a homomorphism corresponding to identification of all z_i except

n into m q-tuples. The variety $\tilde{\mathcal{D}}_{n,m,\theta} = Spec\,\mathcal{N}_{n,m,\theta}$ in this case can be interpreted as the stratum in the discriminant (known also as *coincident root locus*) consisting of the polynomials of degree N with all but n roots having the same multiplicity q. In particular, if $m = 1$ we have the polynomials with a multiple root of multiplicity at least q. The corresponding ideal (the kernel of ϕ) in this particular case was investigated by B. Feigin, Jimbo, Miwa and Mukhin in [11]. They showed that it is also spanned by the Jack polynomials but the geometry of the corresponding Young diagrams is much more complicated. This important paper shows that the case of special values of θ is actually very interesting and deserves more investigation (see [24] for the latest development in this direction). As we have already mentioned to describe the algebraic equations of the discriminant strata is a classical problem which is still largely open [21, 22, 23].

Corollary. *Let $f \in \Lambda_\theta$ be a shifted symmetric function and $\mathcal{L}^f_\theta \in \mathcal{D}_\theta$ be the corresponding quantum integral of the CMS problem. Then for generic θ there exists a quantum integral $\mathcal{L}^f_{n,m,\theta}$ of the deformed CMS problem (17) such that the following diagram is commutative*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\mathcal{L}^f_\theta} & \Lambda \\ \downarrow \varphi & & \downarrow \varphi \\ \Lambda_{n,m,\theta} & \xrightarrow{\mathcal{L}^f_{n,m,\theta}} & \Lambda_{n,m,\theta} \end{array}$$

The super-Jack polynomials (26) are the joint eigenfunctions of all the operators $\mathcal{L}^f_{n,m,\theta}$.

In the next section we investigate the homomorphism: $f \to \mathcal{L}^f_{n,m,\theta}$ in more detail.

6. Shifted symmetric functions and quantum integrals of the deformed CMS problem

Let again $P_{n,m} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ be polynomial algebra in $n+m$ independent variables. The following algebra $\Lambda^\natural_{n,m,\theta}$ introduced in [5] can be considered as a shifted version of the algebra $\Lambda_{n,m,\theta}$. It consists of the polynomials $p(x_1, \dots, x_n, y_1, \dots, y_m)$ which are symmetric in $x_1, \dots, x_n$ and $y_1, \dots, y_m$ separately and satisfy the conditions

$$f(x_i + 1/2, y_j - 1/2) \equiv f(x_i - 1/2, y_j + 1/2)$$

on each hyperplane $x_i + \theta y_j = 0$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. It is easy to check that the *deformed Bernoulli sums*

$$b^*_r(x, y, \theta) = \sum_{i=1}^{n} B_r(x_i + 1/2) + (-\theta)^{r-1} \sum_{j=1}^{m} B_r(y_j + 1/2) \qquad (27)$$

belong to $\Lambda^\natural_{n,m,\theta}$ for all integers $r \geq 0$. According to [5] for generic θ $b^*_r(x, y, \theta)$ generate the algebra $\Lambda^\natural_{n,m,\theta}$.

Now we are going to define the homomorphism $\varphi^\natural$, which is a shifted version of the homomorphism φ from the previous section.

Recall that $H_{n,m}$ denote the set of partitions λ whose diagrams are contained in the fat (n,m)-hook. Consider the following *Frobenius map* $F : H_{n,m} \longrightarrow \mathbb{C}^{n+m} : F(\lambda) = (p_1, \dots, p_n, q_1, \dots, q_m)$, where

$$p_i = \lambda_i - \theta\left(i - \frac{1}{2}\right) - \frac{1}{2}(m - \theta n), \quad q_j = \mu_j^{'} - \theta^{-1}\left(j - \frac{1}{2}\right) + \frac{1}{2}\left(\theta^{-1}m + n\right), \tag{28}$$

and $\mu = (\lambda_{n+1}, \lambda_{n+2}, \dots)$. The motivation for this particular shift comes from the theory of the deformed root systems [5] (see also formula (35) below). The coordinates p_i, q_j can be considered as a version of the Frobenius coordinates on the partitions, so we will call them *modified Frobenius* (n,m)-*coordinates* (cf. [16, 17]).

The image $F(H_{n,m})$ is dense in $\mathbb{C}^{n+m}$ with respect to Zariski topology. The homomorphism

$$\varphi^\natural : \Lambda_\theta \longrightarrow \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_m]$$

is defined by the relation

$$\varphi^\natural(f)(p,q) = f((F^{-1}(p,q)),$$

where $(p,q) \in F(H_{n,m})$. In other words we consider the shifted symmetric function f as a function on the partitions from the fat hook and re-write it in the modified Frobenius (n,m)-coordinates. The fact that as a result we will have a polynomial is not obvious. To prove it consider the following shifted versions of the Bernoulli sums (9):

$$b_k^\natural(z,\theta) = \sum_{i\geq 1}\left[B_k(z_i + \frac{1}{2} + \theta(\frac{1}{2} - i)) - B_k(\frac{1}{2} + \theta(\frac{1}{2} - i))\right] \tag{29}$$

Lemma 4. *The image $\varphi(f)$ of a shifted symmetric function $f \in \Lambda_\theta$ is a polynomial. For the Bernoulli sums $b_k^\natural(z,\theta)$ it can be given by the following explicit formula:*

$$\begin{aligned}\varphi^\natural\left(b_k^\natural(z,\theta)\right) &= \sum_{i=1}^{n}\left[B_k(x_i + \frac{1}{2} + \frac{1}{2}(m - \theta n)) - B_k(\frac{1}{2} + \theta(\frac{1}{2} - i))\right] \\ &+ (-\theta)^{k-1}\sum_{j=1}^{m}\left[B_k(y_i + \frac{1}{2} - \frac{1}{2}(\theta^{-1}m - n)) - B_k(\frac{1}{2} + n + \theta^{-1}(\frac{1}{2} - j))\right]\end{aligned} \tag{30}$$

Proof. Assume that $z_i = \lambda_i$, where $\lambda \in H_{n,m}$. Then we have

$$\begin{aligned}\varphi^\natural\left(b_k^\natural(\lambda,\theta)\right) &= \sum_{i=1}^{n}\left[B_k(\lambda_i + \frac{1}{2} + \theta(\frac{1}{2} - i)) - B_k(\frac{1}{2} + \theta(\frac{1}{2} - i))\right] \\ &+ \sum_{i\geq 1}\left[B_k(\mu_i + \frac{1}{2} + \theta(\frac{1}{2} - n - i)) - B_k(\frac{1}{2} + \theta(\frac{1}{2} - n - i))\right]\end{aligned}$$

The same calculations as in Lemma 1 show that the last sum is equal to

$$(-\theta)^{k-1}\sum_{j=1}^{m}\left[B_k(\mu'_j+\frac{1}{2}+n+\theta^{-1}(\frac{1}{2}-j))-B_k(\frac{1}{2}+n+\theta^{-1}(\frac{1}{2}-j))\right].$$

This proves the formula (30). Since the Bernoulli sums generate Λ_θ this implies the first part of the Lemma as well.

Theorem 3. *If θ is not a negative rational number or zero, then the image of the homomorphism $\varphi^\natural$ coincides with the algebra $\Lambda^\natural_{n,m,\theta}$ and the kernel of $\varphi^\natural$ is spanned by the shifted Jack polynomials $P^*_\lambda(z,\theta)$ corresponding to the Young diagrams which are not contained in the fat (n,m)-hook.*

Proof. The first claim follows from Lemma 4 and Theorem 2 from [5]. To prove the statement about the kernel consider a shifted Jack polynomial $P^*_\lambda(z,\theta)$ with $\lambda \in \bar{H}_{n,m}$. Let μ be a partition whose diagram is contained in the fat (n,m)-hook. Since this implies that the diagram of λ is not a subset of the one of μ according to the Extra Vanishing Property of shifted Jack polynomials (see Section 3) we have $P^*_\lambda(\mu,\theta)=0$. Thus we have shown that $P^*_\lambda(z,\theta)$ with $\lambda \in \bar{H}_{n,m}$ belong to the kernel of φ. To show that they generate the kernel one should note that

$$\varphi^\natural(P^*_\lambda(z,\theta)) = \varphi(P_\lambda(z,\theta))(x_1,\dots,x_n,-\theta y_1,\dots,-\theta y_m)+\dots,$$

where dots mean the terms of degree less than $|\lambda|$. From Theorem 2 it follows that $\varphi^\natural(P^*_\lambda(z,\theta))$ with $\lambda \in H_{n,m}$ are linearly independent. Theorem is proved.

Corollary. *For generic θ the functions*

$$SP^*_\lambda(x,y,\theta) = \varphi^\natural(P^*_\lambda(z,\theta))$$

with $\lambda \in H_{n,m}$ form a basis in $\Lambda^\natural_{n,m,\theta}$.

We will call the polynomials $SP^*_\lambda(x,y,\theta)$ the *shifted super-Jack polynomials.* They have the following independent definition (cf. [9, 10] and section 3).

Let λ and ν be two arbitrary partitions from the fat (n,m)-hook and $(p(\lambda),q(\lambda))$ be the corresponding modified Frobenius (n,m)-coordinates (28), then the shifted super-Jack polynomials SP^*_ν are uniquely determined by the condition $\deg SP^*_\nu \le |\nu|$ and the following property:

$$SP^*_\nu(p(\lambda),q(\lambda),\theta) = \begin{cases} H(\lambda), & \lambda=\nu \\ 0, & |\lambda|\le|\nu|, \lambda\neq\nu \end{cases} \tag{31}$$

where $H(\lambda)$ is the same as above (see formula (4) in section 3).

We have also the following version of the *Extra Vanishing Property for the shifted super-Jack polynomials*:

$$SP^*_\nu(p(\lambda),q(\lambda),\theta)=0 \tag{32}$$

unless the diagram of ν is a subset of the diagram of λ.

Consider now the algebra of differential operators in $n+m$ variables with rational coefficients belonging to $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_m,(x_i-x_j)^{-1},(x_i-y_l)^{-1},(y_k-x_l)^{-1}]$. We denote it as $\mathcal{D}(n,m)$.

Theorem 4. *For generic values of θ there exists a unique monomorphism $\psi:\Lambda^{\natural}_{n,m,\theta}\to\mathcal{D}(n,m)$ such that the following diagram is commutative*

$$\begin{array}{ccc}\Lambda_\theta & \xrightarrow{\chi} & \mathcal{D}(\theta)\\ \downarrow\varphi^{\natural} & & \downarrow res\\ \Lambda^{\natural}_{n,m,\theta} & \xrightarrow{\psi} & \mathcal{D}(n,m)\end{array}$$

where χ is the inverse Harish-Chandra homomorphism and res is the operation of restriction on the generalised discriminant described in the previous section.

Indeed let f be a shifted symmetric function from Λ_θ, $\mathcal{L}^f_\theta$ and $\mathcal{L}^f_{n,m,\theta}=res(\mathcal{L}^f_\theta)$ be the corresponding quantum integrals of CMS and deformed CMS problems. We know that if $P_\lambda(z,\theta)$ is a Jack symmetric function then

$$\mathcal{L}^f_{n,m,\theta}\varphi(P_\lambda(z,\theta))=f(\lambda)\varphi(P_\lambda(z,\theta)).$$

Therefore according to Theorem 2 $\mathcal{L}^f_{n,m,\theta}\equiv 0$ if and only if $f(\lambda)=0$ for any λ with the diagram contained in the fat (n,m)-hook. Now from Theorem 3 it follows that $Ker(res\circ\chi)=Ker\varphi^{\natural}$.

Let us denote by $\mathcal{D}(n,m,\theta)$ the image of homomorphism ψ. We claim that it is generated by the following quantum integrals of the deformed CMS problem $\mathcal{L}_p$ introduced in [5].

It will be convenient for us to change the notation now by introducing two sets of indices: $I_0=\{1,2,\dots,n\}$ and $I_1=\{\bar 1,\bar 2,\dots,\bar m\}$ and put $x_{\bar j}=y_j$. Let p denote the (parity) function on $I=I_0\cup I_1$ such that $p(i)=0, i\in I_0$ and $p(i)=1, i\in I_1$.

Now define by induction the differential operators $\partial_i^{(p)},\quad i\in I$ as follows: for $p=1$

$$\partial_i^{(1)}=(-\theta)^{p(i)}x_i\frac{\partial}{\partial x_i}$$

and for $p>1$

$$\partial_i^{(p)}=\partial_i^{(1)}\partial_i^{(p-1)}-\frac{1}{2}\sum_{j\neq i}(-\theta)^{1-p(j)}\frac{x_i+x_j}{x_i-x_j}\left(\partial_i^{(p-1)}-\partial_j^{(p-1)}\right). \qquad (33)$$

The differential operators $\mathcal{L}_p$ are defined as the sum

$$\mathcal{L}_p=\sum_{i\in I}(-\theta)^{-p(i)}\partial_i^{(p)}. \qquad (34)$$

Theorem 5. *The operators $\mathcal{L}_p$ for all $p=1,2,\dots$ map the algebra $\Lambda_{n,m,\theta}$ into itself.*

The proof is based on the following two technical lemmas. Let us denote by $J_{1,\bar 1}$ the ideal in the polynomial algebra $\mathbb{C}[x_1,\dots,x_n,x_{\bar 1},\dots,x_{\bar m}]$ generated by $x_1-x_{\bar 1}$.

Lemma 5. *The following operators*

1) $\partial_1^{(1)} - \theta^{-1}\partial_{\bar 1}^{(1)}$

2) $\partial_1^{(1)} + \partial_{\bar 1}^{(1)} - \frac{1}{2}(\theta+1)\frac{x_1+x_{\bar 1}}{x_1-x_{\bar 1}}$

3) $\partial_1^{(1)}\partial_{\bar 1}^{(1)} + \frac{1}{2}(-\theta\partial_1^{(1)} - \partial_{\bar 1}^{(1)})\frac{x_1+x_{\bar 1}}{x_1-x_{\bar 1}}$

4) $(\partial_1^{(1)})^2 - \theta^{-1}(\partial_{\bar 1}^{(1)})^2 - (\partial_1^{(1)} - \partial_{\bar 1}^{(1)})\frac{x_1+x_{\bar 1}}{x_1-x_{\bar 1}}$

map the ideal $J_{1,\bar 1}$ *into itself.*

The proof is straightforward.

Lemma 6. *The following operators*

1) $\partial_1^{(p)} - \partial_{\bar 1}^{(p)}$

2) $(\partial_1^{(1)} - \partial_{\bar 1}^{(1)})\partial_i^{(p)}$, $i \neq 1, \bar 1$

3) $\partial_1^{(1)}\partial_{\bar 1}^{(p)} - \partial_{\bar 1}^{(1)}\partial_1^{(p)}$

4)$(\partial_1^{(1)} - \partial_{\bar 1}^{(1)})(\partial_1^{(p-1)} - \theta^{-1}\partial_{\bar 1}^{(p-1)})$ *map the algebra* $\Lambda_{n,m,\theta}$ *into the ideal* $J_{1,\bar 1}$. *If* $f \in \Lambda_{n,m,\theta}$ *then* $\partial_i^{(p)} f$ *is a polynomial.*

Proof is by induction with the use of Lemma 5.

To prove Theorem 5 consider any polynomial $f \in \Lambda_{n,m,\theta}$. According to Lemma 6 $\mathcal{L}_p(f)$ is also a polynomial. To show that it belongs to $\Lambda_{n,m;\theta}$ we have to check that $(\partial_1^{(1)} - \partial_{\bar 1}^{(1)})\,(\mathcal{L}_p)\,(f)$ belongs to the ideal $J_{1,\bar 1}$. But

$$(\partial_1^{(1)}-\partial_{\bar 1}^{(1)})\,(\mathcal{L}_p)\,(f) = \sum_{i\neq 1,\bar 1}(\partial_1^{(1)}+\partial_{\bar 1}^{(1)})(-\theta)^{-p(i)}\partial_i^{(p)}(f)+(\partial_1^{(1)}+\partial_{\bar 1}^{(1)})(\partial_1^{(p)}-\theta^{-1}\partial_{\bar 1}^{(p)})(f),$$

so due to Lemma 6 this is true since all the summands belong to $J_{1,\bar 1}$. Theorem 5 is proved.

Now according to [5] the operators $\mathcal{L}_p$ commute with each other and in particular with $\mathcal{L}_2$, which is the deformed CMS operator. By the standard arguments (see e.g. proof of Theorem 1 from [5]) one can show that they must commute also with the operators $\mathcal{L}^f_{n,m,\theta}$ for any $f \in \Lambda^\natural_{n,m,\theta}$.

Since the operators $\mathcal{L}^f_{n,m,\theta}$ separate the super-Jack polynomials $SP_\lambda(x,y,\theta)$ with $\lambda \in H_{n,m}$ these polynomials must be also the eigenfunctions for the operators $\mathcal{L}_p$. For any $\lambda \in H_{n,m}$ define $\nu_\lambda = (\lambda_1,\dots,\lambda_n,\mu'_1,\dots,\mu'_m)$, where $\mu = (\lambda_{n+1},\lambda_{n+2},\dots)$. It follows from the results of [5] that

$$\mathcal{L}_p SP_\lambda(x,y,\theta) = \mathcal{Z}_p(\nu_\lambda - \rho)SP_\lambda(x,y,\theta)$$

where

$$\rho_i = \theta(i-\frac{1}{2}) + \frac{1}{2}(m-\theta n), \quad \rho_{n+j} = \theta^{-1}(j-\frac{1}{2}) - \frac{1}{2}(\theta^{-1}m+n), \qquad (35)$$

$i = 1,\dots,n,\ j = 1,\dots,m$ and

$$\mathcal{Z}_p(\lambda) = \lambda_1^p + \dots + \lambda_n^p + (-\theta)^p(\mu_1^p + \dots + \mu_m^p) + \dots,$$

where dots mean the terms of less lexicographic order. It was also proven in [5] that for generic θ $\mathcal{Z}_p$ generate the algebra $\Lambda^\natural_{n,m,\theta}$. This means that one

can express any operator $\mathcal{L}^f_{n,m,\theta}$ as a polynomial of the operators $\mathcal{L}_p$. Thus we have proved the following

Theorem 6. *For generic values of θ the operators $\mathcal{L}_p$ for $p = 1, 2, \dots$ generate the algebra $\mathcal{D}(n, m, \theta)$.*

7. Filters and CMS-invariant ideals in Λ

The previous results lead naturally to the following question: which ideals I in Λ are invariant under the action of CMS-operator L_θ and its quantum integrals ? To answer this question it is useful to introduce the following notion which was probably first used by Knop and Sahi [9], but we will follow here the terminology of the recent paper by Regev [27].

Let $\mathcal{P}$ be the set of all partitions (or Young diagrams).

Definition. The subset $\Omega \subset \mathcal{P}$ is called *filter* if it is closed under the inclusion. In other words Ω is a filter if for any diagram $\lambda \in \Omega$ and any μ such that $\lambda \subset \mu$ it follows that $\mu \in \Omega$.

Theorem 7. *For generic θ (more precisely, for θ not a nonpositive rational) there is a bijection between the set of CMS-invariant ideals in Λ and the set of filters.*

Proof. CMS-invariance implies that the corresponding ideal I is a linear span of certain Jack polynomials $P_\lambda = P_\lambda(\theta)$:

$$I = Span(P_\lambda(\theta), \lambda \in \Omega_I)$$

for some $\Omega_I \subset \mathcal{P}$. Let us prove that Ω_I is a filter. Take any $\lambda \in \Omega_I$ and $\lambda \subset \mu$. We should show that $P_\mu \in I$. When $\mu - \lambda$ is one box then this follows from Pieri formula [6, 7]

$$P_\lambda P_1 = \sum_\nu \psi_{\nu/\lambda}(\theta) P_\nu,$$

where the sum is taken over partitions ν obtained by adding one box to λ. If j is such that $\nu_j = \lambda_j + 1$ then $\psi_{\nu/\lambda}(\theta)$ is given by

$$\psi_{\nu/\lambda}(\theta) = \prod_{i=1}^{j-1} \frac{((j-i-1)\theta + \lambda_i - \lambda_j)((j-i+1)\theta + \lambda_i - \lambda_j - 1)}{((j-i)\theta + \lambda_i - \lambda_j - 1)((j-i)\theta + \lambda_i - \lambda_j)}$$

Since θ is not a negative rational number or zero all coefficients $\psi_{\nu/\lambda}(\theta)$ are nonzero. Now CMS-invariance of I implies that all P_ν in this sum belong to I. By induction the same is true for all ν, such that $\lambda \subset \nu$.

Now let Ω be a filter. Let us show that

$$J = Span\,(P_\lambda(\theta), \lambda \in \Omega)$$

is an ideal in Λ. It is enough to show that $P_\lambda e_r \in J$, where e_r is elementary symmetric function. But this is a direct consequence of the Pieri formula

$$P_\lambda e_r = \sum_\nu \psi_{\nu/\lambda}(\theta) P_\nu$$

where now the sum is taken over partitions ν such that $\lambda \subset \nu$ and $\nu - \lambda$ is a vertical r-strip. The coefficients $\psi_{\nu/\lambda}(\theta)$ have the following form (see [6], VI.6):

$$\psi_{\nu/\lambda}(\theta) = \prod_{s \in C_{\nu/\lambda} - R_{\nu/\lambda}} \frac{b_\nu(s)}{b_\lambda(s)}$$

where

$$b_\lambda(s) = \frac{c'_\theta + \theta}{c'_\theta + 1}, \quad if \quad s \in \lambda, \quad and \quad 1 \quad otherwise$$

and $C_{\nu/\lambda}$ (resp. $R_{\nu/\lambda}$) denote the union of the columns (resp. rows) of ν that intersect ν/λ and c'_θ are given by the formula (5) above. Note that our assumption on θ implies that $c'_\theta + 1 \neq 0$. Theorem 7 now follows.

We will denote the ideal $I = Span(P_\lambda(\theta), \lambda \in \Omega)$ corresponding to filter Ω as $I_\Omega(\theta)$.

Let $\Omega(\lambda^{(1)}, \dots, \lambda^{(n)})$ be the set of Young diagrams μ such that μ contains at least one of the Young diagrams $\lambda^{(1)}, \dots, \lambda^{(n)}$. It is easy to see that $\Omega(\lambda^{(1)}, \dots, \lambda^{(n)})$ is a filter. We will call such a filter *finitely generated.*

Lemma 7 (A. Regev [27]**).** *Any filter is finitely generated.*

This result was proven by Regev [27] using combinatorial arguments. We will give a simpler algebraic proof using our results from [5].

Proof of Lemma 7. Any filter contains some rectangle π. Consider the ideal $I_\pi(\theta) = Span(P_\lambda(\theta), \pi \subset \lambda)$ assuming that θ is generic. It is obviously contained in $I_\Omega(\theta)$. According to [5] for generic θ the algebra $\Lambda/I_\pi(\theta)$ is finitely generated (see Theorem 5 in [5]).

Therefore the ideal $I_\Omega(\theta)/I_\pi(\theta) \subset \Lambda/I_\pi(\theta)$ is finitely generated as well (as an ideal in Noetherian algebra). Let $P_{\lambda^{(1)}}(\theta), \dots, P_{\lambda^{(n)}}(\theta)$ be Jack polynomials whose images generate $I_\Omega(\theta)/I_\pi(\theta) \subset \Lambda/I_\pi(\theta)$. We claim that $\Omega = \Omega(\pi, \lambda^{(1)}, \dots, \lambda^{(n)})$. Let $\lambda \in \Omega$ then $P_\lambda \in I_\Omega$. By assumption there exist $f_1, \dots, f_n \in \Lambda$ such that

$$P_\lambda - \sum_{i=1}^{n} f_i P_{\lambda^{(i)}} \in I_\pi$$

Now using again Pieri formula we conclude that P_λ is a linear combination of P_ν where ν contains either $\lambda^{(i)}$ or π. Lemma is proved.

Let now Ω be a filter and consider the corresponding algebra $\Lambda_\Omega(\theta) = \Lambda/I_\Omega(\theta)$.

Theorem 8. *For any filter Ω and generic θ the algebra $\Lambda_\Omega(\theta)$ is finitely generated.*

Proof. Consider a rectangle $\pi \in \Omega$ and the corresponding ideal

$$I_\pi(\theta) = Span(P_\lambda(\theta), \lambda \supset \pi) \subset \ I_\Omega(\theta)$$

Since $\Lambda_\pi(\theta) = \Lambda/I_\pi(\theta)$ for generic θ is finitely generated the algebra $\Lambda_\Omega(\theta)$ is also finitely generated as a homomorphic image of $\Lambda_\pi(\theta)$.

Consider now the special case when $\Omega = \Omega(\lambda)$ is generated by one Young diagram λ. If λ is $(n+1) \times (m+1)$ rectangle then for generic θ according to

[5] $\Lambda_\lambda(\theta) = \Lambda_{n,m,\theta}$. Note that when $m = 0$ we have the standard symmetric polynomial algebra Λ_n. The question is what happens for non-rectangular Young diagrams.

Recall that an algebra is called a *domain* if it has no zero divisors , i. e. $ab = 0$ implies that either $a = 0$ or $b = 0$. An algebra is called *nilpotent free* if $a^n = 0$ implies $a = 0$.

Theorem 9. *For generic θ the algebra $\Lambda_{\Omega(\lambda)}(\theta)$ is nilpotent free. It is a domain if and only if λ is a rectangle. The ideal $I_{\Omega(\lambda)}(\theta)$ is the intersection of the prime ideals $I_{\pi_i}(\theta)$, corresponding to the maximal rectangular subdiagrams $\pi_1, \dots, \pi_k \subset \lambda$.*

Proof. First of all if λ is a rectangle π this follows from the results of [5] and Theorem 2 above since the corresponding algebra $\Lambda_\pi(\theta)$ can be realised as a subalgebra $\Lambda_{n,m,\theta}$ in the polynomial algebra. Let now λ be arbitrary. Consider all maximal rectangular subdiagrams $\pi_1, \dots, \pi_k \subset \lambda$, then

$$I_{\Omega(\lambda)}(\theta) = \bigcap_{i=1}^{k} I_{\pi_i}(\theta),$$

where each of the ideals $I_{\pi_i}(\theta)$ is prime. In particular, $P_{\pi_i}(\theta) \notin I_{\Omega(\lambda)}(\theta)$ but the product $P_{\pi_1}(\theta) \dots P_{\pi_k}(\theta) \in I_{\Omega(\lambda)}(\theta)$. Thus $\Lambda_{\Omega(\lambda)}(\theta)$ contains zero divisors unless λ is rectangular.

Show now that it is nilpotent free. Take any $a \in \Lambda$ then if $a^n \in I_{\Omega(\lambda)}(\theta)$ then $a^n \in I_{\pi_i}(\theta)$ for all $i = 1, \dots, k$. But since $I_{\pi_i}(\theta)$ are prime this implies that $a \in I_{\pi_i}(\theta)$ and hence $a \in I_{\Omega(\lambda)}(\theta)$, which completes the proof.

Corollary. *Affine algebraic variety*

$$\mathcal{M}_\lambda(\theta) = Spec\, \Lambda_{\Omega(\lambda)}(\theta)$$

corresponding to a non-rectangular Young diangram λ is reducible. Its irreducible components are the generalised discriminants related to maximal rectangular subdiagrams of λ.

Let us discuss now the filters generated by two Young diagrams λ_1 and λ_2. We restrict ourselves by the case when $\lambda_1 = \pi_1$, $\lambda_2 = \pi_2$ are rectangular.

Let $\pi = \pi_1 \cap \pi_2$ be the intersection of these rectangles. We will assume that π is different from π_1 and π_2.

Theorem 10. *If π_1 and π_2 are two rectangular Young diagrams such that neither of them contains another then for generic θ the algebra $\Lambda_{\Omega(\pi_1,\pi_2)}(\theta)$ has nilpotent elements.*

Proof. Let the intersection π of $\pi_1 \cap \pi_2$ be of size $n \times m$. Consider the following polynomial

$$R_{n,m} = \prod_{i,j} (x_i - y_j)^2$$

It is easy to see that $R_{n,m} \in \Lambda_{n,m,\theta}$ for any θ. One can represent it as a sum

$$R_{n,m} = \sum_\lambda c_\lambda(\theta) SP_\lambda(x, y, \theta)$$

where λ contains the $n \times m$ rectangle. Indeed, under the homomorphism sending x_n and y_n to zero $R_{n,m}$ obviously becomes zero. This means that the corresponding $P_\lambda(\theta)$ belong to the kernel of

$$\varphi_{n-1,m-1} : \Lambda \longrightarrow \Lambda_{n-1.m-1}$$

which is known to consist of $P_\lambda(\theta)$ with λ containing $n \times m$ rectangle. Consider now the smallest rectangle π^* which contains both π_1 and π_2. If π_1 and π_2 are $n \times M$ and $N \times m$ rectangles respectively, then π^* has the size $N \times M$. We claim that

$$R_{N,M} x_1 \dots x_N = \sum_{\lambda \supset \Pi_{N,M}} c_\lambda SP_\lambda(x, y, \theta) \tag{36}$$

as elements of $\Lambda_{N,M,\theta}$. Indeed

$$R_{N,M+1} = \sum_{\lambda \supset \Pi_{N,M+1}} c_\lambda SP_\lambda(x, y, \theta)$$

in $\Lambda_{N,M+1,\theta}$. Putting $y_{M+1} = 0$ we came to (36). Now take the natural homomorphism

$$\Lambda_{N,M,\theta} \longrightarrow \Lambda_{\Omega(\pi_1,\pi_2)}(\theta).$$

Since the right hand side of (36) becomes zero we come to the relation

$$R_{N,M} x_1 \dots x_N = 0$$

in $\Lambda_{\Omega(\pi_1,\pi_2)}(\theta)$. Similarly one has the relation

$$R_{N,M} y_1 \dots y_M = 0$$

in $\Lambda_{\Omega(\pi_1,\pi_2)}(\theta)$. We claim that this implies that $R^2_{N,M} = 0$. Indeed $R_{N,M}$ is the sum of monomials each of them is divisible either by the product $x_1 \dots x_N$ or $y_1 \dots y_M$. Theorem 10 is proved.

Conjecture. *For generic θ the radical of the ideal $I_{\Omega(\pi_1,\pi_2)}(\theta)$ is equal to $I_{\Omega(\pi)}(\theta)$, where $\pi = \pi_1 \cap \pi_2$.*

Recall that the *radical* $R(I)$ of the ideal $I \subset \Lambda$ consists of the elements $a \in \Lambda$ such that $a^n \in \Lambda$ for some n. It might help to prove this conjecture if the following *Stanley conjecture* about Jack polynomials would be true: if

$$P_\lambda(\theta) P_\mu(\theta) = \sum_\nu c^\nu_{\lambda\mu}(\theta) P_\nu(\theta)$$

is the expansion of the product of two Jack polynomials then the coefficient $c^\nu_{\lambda\mu}(\theta) \neq 0$ if and only if $c^\nu_{\lambda\mu}(1) \neq 0$ (see Conjecture 8.4 in [7]). Unfortunately as far as we know it is still an open question.

Thus we see that rectangular Young diagrams (and related algebras $\Lambda_{n,m,\theta}$) play a very special role in the theory of filters and corresponding algebras $\Lambda_\Omega(\theta)$. This shows a fundamental importance of the generalised discriminants and gives another justification for our investigation.

8. Combinatorial formulas

In this section we give some combinatorial formulas for the super-Jack polynomials and shifted super-Jack polynomials generalising the results by Stanley, Okounkov and Olshanski (see [7, 10, 12]) Let us recall these results.

A tableau T on λ is called a *reverse tableau* if its entries strictly decrease down the columns and weakly decrease in the rows. By $T(\square)$ denote the entry in the square $\square \in \lambda$. The following combinatorial formula for shifted Jack polynomial was proved by Okounkov in [12]:

$$P^*_\lambda(x,\theta) = \sum_T \varphi_T(\theta) \prod_{\square\in\lambda} \left(x_{T(\square)} - c_\theta(\square)\right) \tag{37}$$

where for a square $\square = (i,j)$

$$c_\theta(\square) = (j-1) - \theta(i-1) \tag{38}$$

(see formula (2.4) in [10]). Here the sum is taken over all reverse tableaux on λ with entries in $\{1,2,\dots\}$ and $\varphi_T(\theta)$ is the same weight of tableau as in the combinatorial formula for ordinary Jack polynomials [7], [6]:

$$P_\lambda(x,\theta) = \sum_T \varphi_T(\theta) \prod_{\square\in\lambda} x_{T(\square)}. \tag{39}$$

We should mention that in [6] sum in formula (39) is taken over ordinary tableaux but since $P_\lambda(x,\theta)$ is symmetric it also holds if the sum in the right-hand side is taken over all reverse tableaux. We have also the following generalisation of (39) for the skew Jack polynomials

$$P_{\lambda/\mu}(x,\theta) = \sum_T \varphi_T(\theta) \prod_{\square\in\lambda/\mu} x_{T(\square)} \tag{40}$$

where the sum is taken over all reverse tableaux of shape λ/μ with entries in $\{1,\dots,\}$ (see [6]).

Let us consider now a reverse *bitableau* T of type (n,m) and shape λ . We can view T as a filling of a Young diagram λ by symbols $1 < 2 \cdots < n < 1' < 2' \cdots < m'$ such that its entries weakly decrease down the columns and right the rows, besides entries $1, 2 \dots, n$ strictly decrease down the columns and entries $1', 2' \dots, m'$ strictly decrease in rows. Let T_1 be a subtableau in T containing all symbols $1', 2' \dots, m'$ and $T_0 = T - T_1$.

Theorem 11. *For generic values of the parameter θ the super-Jack polynomials can be written as*

$$SP_\lambda(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) = \sum_T \varphi_T(\theta) \prod_{\square\in\lambda} x_{T(\square)} \tag{41}$$

where $x_{j'}$ is denoted as y_j and

$$\varphi_T(\theta) = (-1)^{|\mu|} \varphi_{T_1'}(1/\theta) \varphi_{T_0}(\theta) \frac{H(\mu,\theta)}{\theta^{|\mu|} H(\mu', 1/\theta)}$$

Proof follows directly from the formulas (25),(39),(40).

To formulate a similar result for shifted super-Jack polynomials it will be convenient for us to use instead of the algebra $\Lambda^{\natural}_{n,m,\theta}$ the following algebra $\Lambda^{\flat}_{n,m,\theta}$ consisting of the polynomials $p(x_1,\dots,x_n,y_1,\dots,y_m)$ which are symmetric in $x_i-\theta(i-1), i=1,\dots,n$ and $y_j-\theta^{-1}(j-1), j=1,\dots,m$ separately and satisfy the conditions

$$f(x_i+1,y_j-1)\equiv f(x_i,y_j)$$

on each hyperplane $x_i+\theta(1-i)=\theta(y_j+n-1)+1-j=0$ for $i=1,\dots,n$ and $j=1,\dots,m$. It is easy to check that the shift τ

$$\tau(x_i)=x_i+\rho_i,\quad \tau(y_j)=\ y_j+\rho_{n+j},$$

ρ is given by the formula (35), establishes an isomorphism between $\Lambda^{\flat}_{n,m,\theta}$ and $\Lambda^{\natural}_{n,m,\theta}$.

Consider the homomorphism $\varphi^{\flat}=\tau^{-1}\varphi^{\natural}$

$$\varphi^{\flat}:\Lambda_{\theta}\longrightarrow\Lambda^{\flat}_{n,m,\theta}$$

Recall that $H_{n,m}$ is the set of partitions λ such that $\lambda_{n+1}\le m$ and $\mu=(\lambda_{n+1},\lambda_{n+2},\dots)$. Consider the following map $F^{\flat}:H_{n,m}\longrightarrow\mathbb{C}^{n+m}$: $F^{\flat}(\lambda)=(a_1,\dots,a_n,b_1,\dots,b_m)$ where $a_i=\lambda_i, i=1,\dots,n$ and $b_j=\mu'_j, j=1,\dots,m$. The set $F^{\flat}(H_{n,m})$ is dense in $\mathbb{C}^{n+m}$ with respect to Zariski topology. It is easy to see that

$$\varphi^{\flat}(f)(a,b)=f(F^{\flat-1}(a,b))$$

where $(a,b)\in F^{\flat}(H_{n,m})$ and $f\in\Lambda_{\theta}$.

We are going to present a combinatorial formula for the following version of the shifted super-Jack polynomial

$$SP^{\flat}_{\lambda}(x,y,\theta)=\varphi^{\flat}(P^{*}_{\lambda}(z,\theta)).$$

Recall that a reverse tableau T type (n,m) and shape λ is a filling by symbols $1<2<\dots<n<1'<2'<\dots<m'$ such that

1) in each row (resp. column) of T the symbols decrease in the weak sense from left to right (resp. from top to bottom)

2) there is at most one marked symbol j' in each row and at most one unmarked symbol i in each column.

By $T(\Box)$ denote the entry in the square $\Box\in\lambda$.

Theorem 12. *The following formula is true:*

$$SP^{\flat}_{\lambda}=\sum_{T}\varphi_T(\theta)\prod_{\Box\in\lambda}((-\theta)^{p(T(\Box))}x_{T(\Box)}-c_{\theta}(\Box)),\tag{42}$$

where c_{θ} are given by (38).

Proof. Let us consider the skew diagram λ/μ and define *skew shifted Jack polynomials* by the following natural generalisation of the Okounkov's

formula 37:

$$P^*_{\lambda/\mu}(x,\theta)=\sum_T \varphi_T(\theta)\prod_{\Box\in\lambda/\mu}\left(x_{T(\Box)}-c_\theta(\Box)\right).$$

In [12] Okounkov proved that

$$P^*_\lambda(z_1,z_2\dots,\theta)=\sum_{\mu\prec\lambda}\varphi_{\lambda/\mu}(\theta)\prod_{\Box\in\lambda/\mu}(z_1-c_\theta(\Box))\,P^*_\mu(z_2,z_3\dots,\theta), \qquad (43)$$

where $\mu\prec\lambda$ means $\lambda_{i+1}\le\mu_i\le\lambda_i$ and $\varphi_{\lambda/\mu}(\theta)$ is the same coefficient as in the formula for the ordinary Jack polynomials

$$P_\lambda(z_1,z_2\dots,\theta)=\sum_{\mu\prec\lambda}\varphi_{\lambda/\mu}(\theta)z_1^{|\lambda/\mu|}P_\mu(z_2,z_3\dots,\theta)$$

(see [7] and [6], section VI.10). Applying the formula (43) n times we get

$$P^*_\lambda(z_1,z_2\dots,\theta)=\sum_{\mu\subset\lambda}P^*_{\lambda/\mu}(z_1,z_2,\dots,z_n,\theta)P^*_\lambda(z_{n+1},z_{n+2}\dots,\theta)$$

and thus

$$\varphi^\flat(P^*_\lambda(z_1,z_2\dots,\theta))=\sum_{\mu\subset\lambda}P^*_{\lambda/\mu}(x_1,x_2,\dots,x_n,\theta)\omega^*(P^*_\mu(z_{n+1},z_{n+2}\dots,\theta)).$$

Now using the duality (7) we have

$$\varphi^\flat(P^*_\lambda(z_1,z_2\dots,\theta))=\sum_{\mu\subset\lambda}P^*_{\lambda/\mu}(x_1,x_2,\dots,x_n,\theta)\frac{H(\mu,\theta)}{H(\mu',1/\theta)}P^*_{\mu'}(y_1,y_2\dots,y_m,1/\theta).$$

But according to the formula (37)

$$P^*_{\mu'}(y_1,y_2\dots,y_m,1/\theta)=\sum_{T_1'}\varphi_{T_1'}(1/\theta)\prod_{\Box'\in\mu'}\left(x_{T_1'(\Box')}-c_{1/\theta}(\Box')\right)=$$

$$\sum_{T_1}\varphi_{T_1}(1/\theta)\prod_{\Box\in\mu}\left(x_{T_1(\Box)}+\frac{1}{\theta}c_\theta(\Box)\right)=(-1/\theta)^{|\mu|}\sum_{T_1}\varphi_{T_1}(1/\theta)\prod_{\Box\in\mu}\left((-\theta)x_{T_1(\Box)}-c_\theta(\Box)\right),$$

where T_1' is the reverse tableau conjugate to T_1. Therefore $\varphi^\flat(P^*_\lambda(z_1,z_2\dots,\theta))$ can be rewritten as

$$\sum_{T_0}(-1/\theta)^{|\mu|}\frac{H(\mu,\theta)}{H(\mu',1/\theta)}\varphi_{T_0}(\theta)\prod_{\Box\in\lambda/\mu}\left(x_{T_0(\Box)}-c_\theta(\Box)\right)\varphi_{T_1}(1/\theta)\prod_{\Box\in\mu}\left((-\theta)x_{T_1(\Box)}-c_\theta(\Box)\right)$$

$$=\sum_T\varphi_T(\theta)\prod_{\Box\in\lambda}((-\theta)^{p(T(\Box))}x_{T(\Box)}-c_\theta(\Box)).$$

This completes the proof.

9. Some open questions.

It seems that for the generic values of the parameter θ the general situation is more or less clear now, so the main problems remain for the special values of the parameter θ. Let us mention some of them.

For a given negative rational θ is it true that the kernel of the homomorphism φ is generated by some Jack polynomials ? If yes, what is the geometry/combinatorics of the corresponding Young diagrams ? The best results known so far in this direction can be extracted from the paper [11] (see also [24]).

How to describe the generators of the algebra $\Lambda_{n,m,\theta}$ for special values of θ ? What are the corresponding Poincare series ? For generic θ the answer to the last question was given in [5].

How is this related to the theory of quasi-invariants for the deformed root systems [26] ? In particular, are there interesting extensions of the algebra $\Lambda_{n,m,\theta}$ for special values of θ ? We know that at least for $m = 1$ the answer is positive (see [26] for details).

There are also several important open questions left in the case of generic θ. In particular, as we have already mentioned above (see Remark 1 in section 5) the finding of a convenient set of generators for the algebra $\Lambda_{n,m,\theta}$ and a satisfactory description of the relations between them is still to be done. Another problem is to extend our investigation of the algebras $\Lambda_\Omega(\theta)$ to a general filter Ω.

A natural question also is about the generalisations of our results for the deformed Macdonald-Ruijsenaars operators introduced in [5]. We are planning to discuss this in a separate paper.

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